Absence of Zeros for the Chromatic Polynomial on Bounded Degree Graphs

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Dedicated to B. Bollobás on the occassion of his 60th birthday.

Abstract

In this paper, I give a short proof of a recent result by Sokal, showing that all zeros of the chromatic polynomial $P_G(q)$ of a finite graph G of maximal degree D lie in the disc $|q| < KD$, where K is a constant that is strictly smaller than 8.

1 Introduction

This paper is dedicated to Béla Bollobás on the occasion of his 60th birthday. Béla is a dear friend and wonderful collaborator. The work presented here concerns a new proof of absence of zeros of the chromatic polynomial on a finite graph of bounded degree, which grew out of one of ten CBMS-lectures I gave in Memphis in the early summer of 2002. Béla was the main organizer, and, as always, a great host. I also would like to thank him for encouraging me to write up this work.

For a finite graph G, the the chromatic polynomial P_G is the unique polynomial $P_G(q)$ in q that is equal to the number of proper colorings of G with q colors when q is a positive integer. It was introduced by Birkhoff [6] in 1912, and can be used to express many properties of the graph G. For example, one of the most well-known uses of the chromatic polynomial (and Birkhoff's original motivation) is associated with an unsuccessful attempt to prove the 4-color theorem by showing that in a region of the complex plane containing the point $q = 4$, the chromatic polynomial of a planar graph has no zeros. Of course, the zeros of the polynomial are interesting in their own right, and have been intensely studied in the combinatorics community. Most of the earlier mathematical results on the chromatic polynomial concern real zeros, see, e.g., [6, 7, 40, 43, 44, 25, 41, 45, 18], but but recently the study of complex zeros has also become quite popular [24, 3, 4, 1, 2, 19, 33, 42, 10, 11, 12, 13, 38, 35, 26, 39].

In this note I present a short proof of a recent result by Sokal [38], who showed that the chromatic polynomial on any finite graph of maximal degree D is free of zeros if q lies outside the disc $\{q \in \mathbb{C}\colon |q| \leq \tilde{K}D\}$, where $\tilde{K}$ is positive constant strictly smaller than 8. Sokal's work was motivated by the work of Biggs, Damarell and Sands, who conjectured [4] such a result for D-regular graphs, as well as a question of Brenti, Royle, and Wagner [10], who asked whether it is true that a result of this form holds for arbitrary graphs.

2 Results and Proof Strategy

2.1 Sokal's Theorem

Let $G = (V, E)$ be a finite graph. The *chromatic polynomial* P_G of the graph G is the polynomial

$$P_G(q) = \sum_{E' \subset E} q^{C(E')}(-1)^{|E'|}, \tag{1}$$

where $C(E')$ the number of connected components of the graph $G' = (V, E')$. Note that $P_G(q)$ is the unique polynomial which is equal to the number of proper colorings of G for integer q, an fact which is easy to establish and was already known to Birkhoff.

In this note, I give a new short proof of Sokal's theorem:

Theorem 1 (Sokal) *Let G is be finite graph of maximal degree D, and let*

$$K(a) = \frac{a + e^a}{\log(1 + ae^{-a})}. \tag{2}$$

Then all zeros of P_G lie inside the disk $\{q \in \mathbb{C} : |q| < DK\}$, where $K = \min_{a \geq 0} K(a)$.

Sokal proved the theorem in terms of an *a priori* different constant $\tilde{K} = \min_{a \geq 0} \tilde{K}(a)$, where

$$\tilde{K}(a) = \inf \Big\{ K' \colon \sum_{n=1}^{\infty} \frac{1}{n!} \Big(\frac{ne^a}{K'}\Big)^{n-1} \leq 1 + ae^{-a} \Big\}. \tag{3}$$

It turns out, however, that the two formulations are equivalent; in fact, I will show that $\tilde{K}(a) = K(a)$ for all $a > 0$. Note that our formulation allows us to get easy upper bounds on the constant K. Choosing, e.g., $a = 2/5$ we get that $K \leq K(2/5) = 7.964... < 8$. By contrast, the representation (3) requires quite a bit of rigorous numerical mathematics to establish good upper bounds on $K = \tilde{K}$.

Remark 2 *Having shown that the chromatic polynomial is non-zero, one can define the entropy per vertex as the quantity $s_G(q) = \frac{1}{|V|} \log P_G(q)$. This raises the question under which circumstances the entropy per vertex has a limit as the number of vertices tends to infinity, and whether this limit is analytic in $1/q$. For graphs which are induced subgraphs of an amenable quasi-transitive graph, these questions have been analyzed in [32].*

2.2 Basic Proof Strategy

The main idea of Sokal was to map the chromatic polynomial into the generating function for independent sets on the intersection graph for the subsets of V, i.e., the graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ whose vertex set consists of all finite subsets $\gamma \subset V$, with an edge between γ and γ' whenever $\gamma \cap \gamma' \neq \emptyset$. More precisely, he showed that it is possible to define complex weights $z(\gamma)$ for the vertices of $\mathcal{V}$ such that the chromatic polynomial can be rewritten as

$$P_G(q) = q^{|V|} \sum_{I \subset \mathcal{V}}{}' \prod_{\gamma \in I} z(\gamma), \tag{4}$$

where the sum runs over the independent sets I in $\mathcal{G}$. Having obtain such a representation, Sokal then referred to a powerful theorem by Dobrushin [16]. Dobrushin's theorem states that a sum of the above form is free of zeros, provided there is a set of constants $c(\gamma) \in [0, \infty)$, $\gamma \in \mathcal{V}$, such that $z(\cdot)$ lies in the polydisc defined by the condition

$$|z(\gamma)| \leq (1 - e^{-c(\gamma)}) \prod_{\substack{\gamma' \in \mathcal{V} \\ \gamma\gamma' \in \mathcal{E}}} e^{-c(\gamma')} \qquad \forall \gamma \in \mathcal{V}. \tag{5}$$

The main technical task was then to show that one can choose the function $c(\cdot)$ in such a way that for $|q| \geq DK$, the weights $z(\cdot)$ obey Dobrushin's condition (5). Sokal based his proof on a lemma which goes back to Rota [34]; applied to the weights $z(\cdot)$, it implies that $|z(\gamma)|$ can be bounded by $|q|^{1-|\gamma|}$ times the number of spanning trees of the induced graph $G[\gamma]$. Using detailed estimates on the number of subtrees $T \subset G$ which have size s and contain a fixed vertex $x \in V$, he then proved that $z(\cdot)$ obeys Dobrushin's condition if $|q| \geq D\tilde{K}(a)$ for some $a > 0$.

The proof of Sokal's theorem in this note also uses Dobrushin's theorem. But it follows a different strategy to verify the condition (5). Instead of using Rota's lemma, it uses an inductive approach which is based on a reduction formula that expresses the weight $z(\gamma)$ in terms of the weights of the subsets $\gamma' \subset \gamma$.

This approach leads to an *a priori* different condition, the condition that $|q| \geq DK(a)$, with $K(a)$ given by (2). When presenting this proof in my lecture in Memphis, I knew that the resulting constant K agreed with Sokal's constant to the accuracy which Sokal had calculated. While I could not verify at the time that K and $\tilde{K}$ were actually equal, I have since found a proof of this fact. Appropriate for this volume, it uses a function that is well-known in random graph theory, where it describes the size of the giant component above the threshold. The equality of K and $\tilde{K}$ therefore provides another example where formulas from random graph theory lead to identities between *a priori* different functions which might be hard to prove without this formula.

3 Preliminaries

3.1 Dobrushin's Theorem

Dobrushin's theorem states that under the condition (5), a sum of the form (4) is non-zero. Dobrushin formulated this theorem in the language of abstract polymer systems. The term *abstract polymer system* was coined in [37], but the mathematical theory of these systems goes back much further [31, 21, 23, 28], and the main ideas are even older [29]. Later developments [15, 30, 20, 14, 22, 27] led to weaker and weaker conditions for the applicability of the theory, and culminated in Dobrushin's work in 1996 [16]. The applications of this theory in mathematical physics are probably as diverse as the applications of the the Lovász Local Lemma [17] are in graph theory and computer science. Interestingly, the connection is not purely sociological: as shown in [36], the *statement* of Dobrushin's theorem and that of the Lovász Local Lemma are equivalent!

In the language of graph theory, an *abstract polymer system* is just a weighted countable graph with complex vertex weights. More explicitly, a pair $(\mathcal{G}, \boldsymbol{z})$ is called an abstract polymer system if $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is a countable graph, and $\boldsymbol{z} : \gamma \mapsto z(\gamma)$ is a complex valued function on $\mathcal{V}$. The vertices of $\mathcal{G}$ are usually called *polymers*, and the complex number $z(\gamma)$ is called the *activity* or *weight* of γ. For a finite subset $\mathcal{U} \subset \mathcal{V}$, one then defines the *partition function*

$$\mathcal{Z}(\mathcal{U}) = {\sum_{I \subset \mathcal{U}}}' \prod_{\gamma \in I} z(\gamma), \tag{6}$$

where the sum goes over independent sets in $\mathcal{G}$.

Dobrushin's theorem gives both the statement that $\mathcal{Z}(\mathcal{U}) \neq 0$, and a bound on the logarithm of $\mathcal{Z}(\mathcal{U})$. In the literature, this statement is usual formulated as a statement about the principal branch of the logarithm, i.e., the version of the logarithm with imaginary part between $-\pi$ and π. Here we follow a slightly different route, and define the logarithm by analytic continuation as follows: assume that $\mathcal{Z}(\mathcal{U}) \neq 0$ inside a polydisk $\mathbb{D}$ of the form

$$|z(\gamma)| \leq R(\gamma) \qquad \text{for all} \qquad \gamma \in \mathcal{U}, \tag{7}$$

where $R(\gamma) \geq 0$ for all $\gamma \in \mathcal{U}$. Since $\mathbb{D}$ is a compact set, we have that $|\mathcal{Z}(\mathcal{U})|$ is bounded from below by a strictly positive constant, implying that we can find a slightly larger open disk $\tilde{\mathbb{D}}$ such that $\mathcal{Z}(\mathcal{U}) \neq 0$ in $\tilde{\mathbb{D}}$. In $\tilde{\mathbb{D}}$, we then define $\log \mathcal{Z}(\mathcal{U})$ by analytic continuation from the intersection of $\tilde{\mathbb{D}}$ with the set $\{\gamma(\cdot)\colon z(\gamma) > 0 \text{ for all } \gamma \in \mathcal{U}\}$. Note that for this version of the logarithms, we have that

$$\log[\mathcal{Z}(\mathcal{U})/\mathcal{Z}(\tilde{\mathcal{U}})] = \log \mathcal{Z}(\mathcal{U}) - \log \mathcal{Z}(\tilde{\mathcal{U}}) \tag{8}$$

whenever $\tilde{\mathcal{U}} \subset \mathcal{U}$, a fact that will be used in our proof of Dobrushin's theorem below.

Theorem 3 (Dobrushin) *Let $(\mathcal{G}, \boldsymbol{z})$ be an abstract polymer system such that the weights $z(\cdot)$ obey the condition (5) for some function $c : \gamma \mapsto c(\gamma)$ from the vertex set $\mathcal{V}$ of $\mathcal{G}$ into $[0, \infty)$. Let $\mathcal{U}$ be a finite set. Then $\mathcal{Z}(\mathcal{U}) \neq 0$, and*

$$\left|\log\bigl(\mathcal{Z}(\mathcal{U})/\mathcal{Z}(\tilde{\mathcal{U}})\bigr)\right| \leq \sum_{\gamma \in \mathcal{U} \setminus \tilde{\mathcal{U}}} c(\gamma) \tag{9}$$

for all $\tilde{\mathcal{U}} \subset \mathcal{U}$.

Proof. We prove the theorem by induction on the size of $\mathcal{U}$. If $|\mathcal{U}| = 0$, i.e., $\mathcal{U} = \emptyset$, we have $\mathcal{Z}(\mathcal{U}) = 1$ and the statements of the theorem are obvious.

Let $n \in \mathbb{N}$ and assume that the statements of the theorem hold for all $\mathcal{U} \subset \mathcal{V}$ with $|\mathcal{U}| \leq n$. Consider a set $\mathcal{U}$ with $|\mathcal{U}| = n + 1$, and let $\tilde{\mathcal{U}}$ be a strict subset of $\mathcal{U}$. Choose $\gamma_0 \in \mathcal{U}$ in such a way that $\gamma_0 \notin \tilde{\mathcal{U}}$. Decomposing the sum representing $\mathcal{Z}(\mathcal{U})$ into a sum over independent sets I not containing the polymer γ_0 and a sum over independent sets containing γ_0, we now rewrite $\mathcal{Z}(\mathcal{U})$ as

$$\mathcal{Z}(\mathcal{U}) = \mathcal{Z}(\mathcal{U}') + z(\gamma_0)\mathcal{Z}(\mathcal{U}_0) = \mathcal{Z}(\mathcal{U}')\Bigl(1 + z(\gamma_0)\frac{\mathcal{Z}(\mathcal{U}_0)}{\mathcal{Z}(\mathcal{U}')}\Bigr) \tag{10}$$

where $\mathcal{U}' = \mathcal{U} \setminus \{\gamma_0\}$ and $\mathcal{U}_0 = \{\gamma \in \mathcal{U}' \colon \gamma\gamma_0 \notin \mathcal{E}\}$. By the bound (5), the inductive assumption (9) and the observation that $\mathcal{U}' \setminus \mathcal{U}_0 = \{\gamma \in \mathcal{U}' \colon \gamma\gamma_0 \in \mathcal{E}\} \subset \{\gamma \in \mathcal{V} \colon \gamma\gamma_0 \in \mathcal{E}\}$ we have

$$\left|z(\gamma_0)\frac{\mathcal{Z}(\mathcal{U}_0)}{\mathcal{Z}(\mathcal{U}')}\right| \leq (1 - e^{-c(\gamma_0)}) \prod_{\substack{\gamma \in \mathcal{V}: \\ \gamma\gamma_0 \in \mathcal{E}}} e^{-c(\gamma)} \exp\Biggl(\sum_{\substack{\gamma \in \mathcal{U}': \\ \gamma\gamma_0 \in \mathcal{E}}} c(\gamma)\Biggr) \leq (1 - e^{-c(\gamma_0)}) < 1. \tag{11}$$

Combined with (10) and the fact that $\mathcal{Z}(\mathcal{U}') \neq 0$ by the inductive assumption, this clearly gives $\mathcal{Z}(\mathcal{U}) \neq 0$. To prove the estimate (9), we rewrite $\log(\mathcal{Z}(\mathcal{U})/\mathcal{Z}(\tilde{\mathcal{U}}))$ as $\log(\mathcal{Z}(\mathcal{U})/\mathcal{Z}(\mathcal{U}')) + \log(\mathcal{Z}(\mathcal{U}')/\mathcal{Z}(\tilde{\mathcal{U}}))$. To bound the first term, we use (11) and (10) once more. Together with the observation that $|\log(1 + y)| \leq -\log(1 - |y|)$ whenever $|y| < 1$, this gives

$$\left|\log(\mathcal{Z}(\mathcal{U})/\mathcal{Z}(\mathcal{U}'))\right| \leq -\log\Bigl(1 - \bigl(1 - e^{-c(\gamma_0)}\bigr)\Bigr) = c(\gamma_0). \tag{12}$$

Bounding $|\log(\mathcal{Z}(\mathcal{U}')/\mathcal{Z}(\tilde{\mathcal{U}}))|$ with the help of the inductive assumption (9), this gives the desired bound on $|\log(\mathcal{Z}(\mathcal{U})/\mathcal{Z}(\tilde{\mathcal{U}}))|$.

□

The statement of Dobrushin's theorem clearly implies that $\log \mathcal{Z}(\mathcal{U})$ is analytic in the interior of the disk defined by (5), allowing one to expand $\log \mathcal{Z}(\mathcal{U})$ into an absolutely convergent Taylor series about $z(\cdot) \equiv 0$. For applications in statistical physics, one usually needs explicit expressions for the coefficients of this expansion; most treatments of abstract polymer systems therefore include a calculation of these coefficients, see, e.g., [37, 14, 22]. But for the application at hand, we only need the fact that under the condition (5), the partition function $\mathcal{Z}(\mathcal{U})$ is free of zeros.

3.2 A graph theoretic lemma

As we will see in the next section, the chromatic polynomial of a graph $G = (V, E)$ can be rewritten in terms of an abstract polymer system with polymers consisting of subsets $\gamma \subset V$ with two or more elements, and weights involving a certain graph function $\phi_c(\cdot)$. In this subsection, we derive an inductive expression for this function, see Lemma 4 below.

Given a non-empty, finite graph $G = (V, E)$, let

$$\phi_c(G) = \sum_{E' \subset E}{}' (-1)^{|E'|}, \tag{13}$$

where the sum $\sum'$ goes over sets $E' \subset E$ such that the graph (V, E') is connected. If G is the empty graph we define $\phi_c(G) = 0$. Note that $\phi_c(G) = 1$ if G is a graph with a single vertex, since in this case the subgraph $(V, \emptyset)$ is a connected spanning graph. On the other hand $\phi_c(G) = 0$ if G is not connected, in accordance with the usual convention that an empty sum is considered to be zero.

To my knowledge the following lemma first appeared in [9]. It is somewhat reminiscent of Rota's Möbius lemmas [34] and certain lemmas of mathematical physics [14] relating "connected" and "disconnected" diagrams, but is also quite different from these earlier lemmas in that it expresses ϕ_c as a sum of terms involving ϕ_c alone.

Lemma 4 *Let $G = (V, E)$ be a non-empty finite graph, let $v \in V$, and let $V_0 = V \setminus \{v\}$. Then*

$$\phi_c(G) = \sum_{\pi \text{ of } V_0} \prod_{Y \in \pi} \big(-\phi_c(G[Y]) I(v \sim Y)\big), \tag{14}$$

where the sum goes over all partitions π of V_0 into non-empty subsets and $I(v \sim Y)$ is the indicator function of the event that there exists at least one vertex $w \in Y$ such that $vw \in E$.

Proof. Let $E' \subset E$ be a set of edges contributing to the right hand side of (13). Let E'_0 be the set of those edges in E' which do not contain the vertex v as an endpoint, and let $G_0 = (V_0, E'_0)$. The connected components of G_0 then induce a partition π of V_0. Summing over all $E' \subset E$ that induce a given partition π, we get a contribution that can be decomposed as the product

$$\prod_{Y \in \pi} \Big[\phi_c(G[Y]) \sum_{\emptyset \neq E''_Y \subset E_Y} (-1)^{|E''_Y|}\Big], \tag{15}$$

where E_Y denotes the set of edges in E that join Y to the vertex v. Observing that

$$\sum_{\emptyset \neq E''_Y \subset E_Y} (-1)^{|E''_Y|} = (1-1)^{|E_Y|} - 1 = -1 \tag{16}$$

if $E_Y \neq \emptyset$, while $\sum_{\emptyset \neq E''_Y \subset E_Y} (-1)^{|E''_Y|} = 0$ if $E_Y = \emptyset$, we obtain (14). $\square$

4 Proof of Theorem 1

In this section, we prove Theorem 1. Following Sokal, we first show that the chromatic polynomial can be rewritten in terms of an abstract contour system. In a second step, we then verify the condition (5). It is here that our approach is different from that of Sokal.

4.1 Mapping to a Polymer System

We start from the representation (1), which we repeat here for the convenience of the reader,

$$P_G(q) = \sum_{E' \subset E} q^{C(E')} (-1)^{|E'|}. \tag{17}$$

Consider a set of edges E' contributing to the right hand side. The connected components of the graph $G' = (V, E')$ then induce a partition of the vertex set V into $C(E')$ disjoint sets $Y_1, \ldots, Y_{C(E')}$. Consider the sum over all spanning subgraphs $G' = (V, E')$ that lead to the same partition π. Observing that the factor $q^{C(E')}$ can be rewritten as $q^{|\pi|}$ where $|\pi|$ is the number of elements of the partition π, this allows us to rewrite $P_G(q)$ as

$$P_G(q) = \sum_{\pi \text{ of } V} \prod_{\gamma \in \pi} \big(q \phi_c(G[\gamma]) \big), \tag{18}$$

where the sum goes over partitions of V into connected subsets, and $G[\gamma]$ is the induced graph on γ. Extracting a factor $q^{|V|}$ and observing that $\phi_c(G[\gamma]) = 1$ if $|\gamma| = 1$, we get

$$P_G(q) = q^{|V|} \sum_{\pi \text{ of } V} \prod_{\substack{\gamma \in \pi: \\ |\gamma| \geq 2}} \big(q^{1-|\gamma|} \phi_c(G[\gamma]) \big). \tag{19}$$

This is the desired representation in terms of an abstract polymer system. Indeed, let $\mathcal{V}$ be the set of all connected sets $\gamma \subset V$ with $|\gamma| \geq 2$, with an edge between $\gamma \subset V$ and $\gamma' \subset V$ whenever $\gamma \cap \gamma' \neq \emptyset$. Let $\mathcal{E}$ be the set of such edges, and $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. The sum in (19) can then be rewritten as a sum over independent sets. Defining the activity of a set $\gamma \in \mathcal{V}$ as

$$z(\gamma) = q^{1-|\gamma|} \phi_c(G[\gamma]), \tag{20}$$

this gives the representation (4) for the chromatic polynomial.

4.2 Verification of Dobrushin's Condition

Given the representation (4), Dobrushin's theorem implies that $P_G(q)$ is free of zeros whenever the condition (5) is satisfied. To verify this condition, it is convenient to rewrite it in a slightly different form.

Let $a : \gamma \mapsto a(\gamma)$ be a function from the set of polymers $\mathcal{V}$ into $[0, \infty)$. Setting $c(\gamma) = \log(1 + |z(\gamma)|e^{a(\gamma)})$, one easily checks that the condition (5) is equivalent to the condition that

$$\sum_{\substack{\gamma \in \mathcal{V}: \\ \gamma\gamma_0 \in \mathcal{E} \\ \text{or } \gamma=\gamma_0}} \log\Big(1 + |z(\gamma)|e^{a(\gamma)}\Big) \leq a(\gamma_0) \tag{21}$$

for all $\gamma_0 \in \mathcal{V}$ with $z(\gamma_0) \neq 0$. Setting $a(\gamma) = a|\gamma|$, where a is a positive real number and $|\gamma|$ denotes the number of elements in γ, it is clearly enough to show that

$$\sum_{\substack{\gamma \in \mathcal{V}: \\ \gamma \ni x}} \log\Big(1 + |z(\gamma)|e^{a|\gamma|}\Big) \leq a \tag{22}$$

for all $x \in V$.

The proof of the next lemma is the main technical step in the proof of Sokal's theorem.

Lemma 5 *Let G be a finite graph of maximal degree D, let $(\mathcal{G}, \boldsymbol{z})$ be the polymer system defined in Section 4.1, and let $a(\gamma) = a|\gamma|$. If $|q| \geq K(a)D$, where $K(a)$ is defined in* (2)*, then*

$$\sum_{\substack{\gamma \in \mathcal{V}: \\ \gamma \ni x}} |z(\gamma)|e^{a|\gamma|} \leq a \tag{23}$$

for all $x \in V$, implying in particular that (22) *and hence the Dobrushin condition* (5) *are satisfied.*

With the help of Dobrushin's theorem, Theorem 1 clearly follows from Lemma 5. The proof of this lemma, and thus the technical meat of our proof of Theorem 1, is based on Lemma 4. By contrast, Sokal's proof is based on an inequality that goes back to Rota, stating that for an arbitrary graph $\tilde{G}$, the absolute value of $\phi_c(\tilde{G})$ is bounded by the number of spanning subtrees of $\tilde{G}$.

Proof of Lemma 5. Setting $\epsilon = e^a|q|^{-1}$ and recalling the definition (20) of $z(\gamma)$, we rewrite the terms on the left hand side of (23) as $|z(\gamma)|e^{a|\gamma|} = e^a\epsilon^{|\gamma|-1}|\phi_c(G[\gamma])|$. Multiplying both sides of (23) by e^{-a} and adding 1 in the form $1 = |\phi_c(G[\{x\}])|\epsilon^{1-1}$ to both sides, we see that the condition (23) is equivalent to the condition

$$\sum_{\substack{\gamma \subset \Lambda: \\ \gamma \ni x}} \epsilon^{|\gamma|-1}|\phi_c(G[\gamma])| \leq 1 + ae^{-a} \tag{24}$$

where the sum runs over all connected subsets of V, including the set $\gamma = \{x\}$ containing only one point.

Given an arbitrary subset $\Lambda \subset V$ and a vertex $x_1 \in V$, let us define

$$F_\Lambda(x_1) = \sum_{\substack{\gamma \subset \Lambda: \\ \gamma \ni x_1}} \epsilon^{|\gamma|-1}|\phi_c(G[\gamma])|. \tag{25}$$

For the proof of the lemma, it is then enough to show

$$F_\Lambda(x_1) \le 1 + ae^{-a} \tag{26}$$

for all $\Lambda \subset V$ and all $x_1 \in \Lambda$.

We prove this bound by induction on the size of Λ. As a first step, we establish the bound

$$F_\Lambda(x_1) \le \exp\Big(\epsilon \sum_{\substack{x_2 \in \Lambda\setminus\{x_1\}: \\ x_1x_2 \in E}} F_{\Lambda\setminus\{x_1\}}(x_2)\Big). \tag{27}$$

To this end, we rewrite the sum in (25) as a sum over sequences of pairwise distinct vertices. Since x_1 is fixed, each set γ of order n corresponds to $(n-1)!$ different sequences, leading to the representation

$$F_\Lambda(x_1) = \sum_{n=1}^{\infty} \frac{\epsilon^{n-1}}{(n-1)!} \sideset{}{'}\sum_{x_2,\dots,x_n \in \Lambda_1} |\phi_c(G(x_1, x_2, \dots, x_n))|, \tag{28}$$

where the sum $\sum'$ goes over sequences of pairwise distinct vertices $x_2,\dots,x_n$ in $\Lambda_1 = \Lambda\setminus\{x_1\}$, and $G(x_1, x_2, \dots, x_n)$ is the graph $G(x_1, x_2, \dots, x_n) = ([n], E(x_1, x_2, \dots, x_n))$, with $[n] = \{1, \dots, n\}$ and $E(x_1, \dots, x_n) = \{ij\colon x_ix_j \in E\}$. Note that we have replaced the induced subgraph $G[\{x_1, x_2, \dots, x_n\}] \subset G$ by its "image" $G(x_1, x_2, \dots, x_n)$ on the vertex set $\{1, \dots, n\}$. This will be convenient when summing over the vertices $x_2, \dots, x_n$, since it decouples the graph structure of the graph $G[\{x_1, x_2, \dots, x_n\}] \subset G$ from the locations of the vertices $x_1, \dots, x_n \in V$.

Using Lemma 4, the right hand side becomes

$$F_\Lambda(x_1) = \sum_{n\ge 1} \frac{1}{(n-1)!} \sideset{}{'}\sum_{x_2,\dots,x_n \in \Lambda_1} \sum_{\pi} \prod_{Y\in\pi} \Big(\epsilon^{|Y|} |\phi_c(G(x_Y))| I(x_1 \sim x_Y)\Big) \tag{29}$$

where the third sum goes over all partitions π of $\{2, \dots, n\}$, x_Y denotes the subsequence of x_2, $\dots$, x_n with indices in Y, and $I(x_1 \sim x_Y)$ is the indicator function of the event that at least one of the vertices in $\{x_i\}_{i\in Y}$ is adjacent to x_1. Relaxing the condition of pairwise distinctness to include only pairwise distinctness within each group $\{x_i\}_{i\in Y}$ and exchanging the sum over partitions with the sum over the vertices $x_2, \dots, x_n$ we then get the bound

$$F_\Lambda(x_1) \le \sum_{n\ge 1} \sum_{\pi} \frac{1}{(n-1)!} \prod_{Y\in\pi} \Big(\epsilon^{|Y|} \sideset{}{'}\sum_{\substack{x_Y \in \Lambda_1^Y: \\ x_1 \sim x_Y,}} |\phi_c(G(x_Y)|\Big). \tag{30}$$

To continue, we note that the last sum on the right hand side only depends on the size of Y, not the particular set $Y \subset \{2, \dots, n\}$. Indeed, relabeling the vertices in the sum over x_Y, and using that a sum over sequences of $|Y|$ distinct vertices in Λ_1 is equal to a sum over subsets of size $|Y|$ times $|Y|!$, we see that the above bound can be rewritten as

$$F_\Lambda(x_1) \le \sum_{n\ge 1} \sum_{\pi} \frac{1}{(n-1)!} \prod_{Y\in\pi} \Big(|Y|! W_{|Y|}\Big), \tag{31}$$

where

$$W_\ell = \sum_{\substack{\gamma\subset\Lambda_1:\\ |\gamma|=\ell,\\ \gamma\sim x_1.}} \epsilon^{|\gamma|}|\phi_c(G([\gamma])|. \tag{32}$$

Next we rewrite the sum over partitions $\pi = \{Y_1, \dots, Y_k\}$ of order k as $\frac{1}{k!}$ times the sum over ordered partitions $(Y_1, \dots, Y_k)$. Using the fact that the number of ordered partitions $\pi = (Y_1, \dots, Y_k)$ of $\{2, \dots, n\}$ with fixed sizes $|Y_1| = n_1, \dots, |Y_k| = n_k$ is equal to

$$\frac{(n-1)!}{n_1! \dots n_k!}, \tag{33}$$

it is then not hard to see that the sums in (31) can be carried out explicitly, leading to the identity

$$\begin{aligned}
\sum_{n\geq 1}\sum_{\pi}\frac{1}{(n-1)!}\prod_{Y\in\pi}\Big(|Y|!W_{|Y|}\Big) &= 1+\sum_{n\geq 1}\sum_{k\geq 1}\sum_{\substack{n_1,\dots,n_k\geq 1:\\ \sum_i n_i=n-1}}\frac{1}{k!}\frac{1}{n_1!\dots n_k!}\prod_{i=1}^{k} n_i!W_{n_i}\\
&= 1+\sum_{k\geq 1}\sum_{n_1,\dots,n_k\geq 1}\frac{1}{k!}\prod_{i=1}^{k}W_{n_i}\\
&= \exp\Big(\sum_{n=1}^{\infty}W_n\Big) = \exp\Big(\sum_{\substack{\gamma\subset\Lambda_1:\\ \gamma\sim x_1}}\epsilon^{|\gamma|}|\phi_c(G[\gamma])|\Big).
\end{aligned} \tag{34}$$

This gives the estimate

$$F_\Lambda(x_1) \leq \exp\Big(\sum_{\substack{\gamma\subset\Lambda_1:\\ \gamma\sim x_1}}\epsilon^{|\gamma|}|\phi_c(G[\gamma])|\Big) \leq \exp\Big(\epsilon\sum_{\substack{x_2\in\Lambda_1:\\ x_2\sim x_1}}\sum_{\substack{\gamma\subset\Lambda_1:\\ \gamma\ni x_2}}\epsilon^{|\gamma|-1}|\phi_c(G[\gamma])|\Big), \tag{35}$$

and hence (27).

With the bound (27) in hand, the proof of (26) is an easy induction argument. Indeed, we clearly have $F_{\{x\}}(x) = 1 \leq 1 + ae^{-1}$. Thus consider a finite subset $\Lambda \subset V$ and some vertex $x_1 \in \Lambda$. Assume by induction that $F_{\Lambda\setminus\{x_1\}}(x_2) \leq 1 + ae^{-1}$ for all $x_2 \in \Lambda_1 = \Lambda \setminus \{x_1\}$. The bound (27) then implies

$$F_\Lambda(x_1) \leq \exp\Big(\epsilon\sum_{\substack{x_2\in\Lambda\setminus\{x_1\}:\\ x_2\sim x_1}}(1+ae^{-1})\Big) \leq \exp\big(\epsilon D(1+ae^{-1})\big). \tag{36}$$

Inserting the value of ϵ and using the bound (2), we have

$$\epsilon D(1+ae^{-1}) = (a+e^a)\frac{D}{|q|} \leq \log(1+ae^{-a}), \tag{37}$$

which proves $F_\Lambda(x_1) \leq 1 + ae^{-a}$, as desired. □

We close this section by proving that our bound and that of Sokal are equivalent. To this end, we consider the function

$$f(x) = \sum_{n=1}^{\infty} \frac{n^{n-1}}{n!} x^n. \tag{38}$$

The radius of convergence for $f(x)$ is clearly $1/e$, and for $x \in (0, 1/e]$, we may express x as ce^{-c} for a uniquely defined $c \in (0, 1]$. Using these two facts, we rewrite Sokal's constant $\tilde{K}(a)$ as

$$\tilde{K}(a) = \frac{e^{a+c(a)}}{c(a)}, \tag{39}$$

where

$$c(a) = \sup\Big\{c \in (0,1]\colon \frac{e^c}{c} f(ce^{-c}) \leq 1 + ae^{-a}\Big\}. \tag{40}$$

To complete the proof, we need a simple fact that is well known in the random graph community (see, e.g., [8], p. 103, eq. (5.6)):

$$f(ce^{-c}) = c \qquad \text{for all} \qquad c \in (0, 1]. \tag{41}$$

As a consequence, $c(a)$ can be calculated explicitly, giving $c(a) = \log(1 + ae^{-a})$ and hence the desired equality of $\tilde{K}(a)$ and $K(a)$.

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